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Harmonic Wavelets towards the Solution of Nonlinear PDE

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Abstract—In this paper, harmonic wavelets, which are analytically defined and band limited, are studied, together with their differentiable properties. Harmonic wavelets were recently applied to the solution of evolution problems and, more generally, to describe evolution operators. In order to consider the evolution of a solitary profile (and to focus on the localization property of wavelets), it seems to be more expedient to make use of functions with limited compact support (either in space or in frequency). The connection coefficients of harmonic wavelets are explicitly computed (in the following) at any order, and characterized by some recursive formulas. In particular, they are functionally and finitely defined by a simple formula for any order of the basis derivatives.

Keywords—Harmonic wavelets, Connection coefficients, Refinable integrals, PDE.

1. INTRODUCTION

The investigation of wavelet solution of (linear and nonlinear) differential problems has been one of the most fruitful applications of wavelet theory. So, if we restrict, in particular, to the Petrov-Galerkin method, wavelet bases were efficiently used to define the solution of PDE equations (see, e.g., [1–7]), integral equations (see, e.g., [8–13]), and more general integrodifferential equations and operators (see, e.g., [14,15]). The many advantages of using wavelets, such as the localization and compression, are combined with their main property, i.e., they (usually) form orthonormal bases (in suitable functional spaces). Thus, wavelets easily fulfill one of the basic requirements of the Petrov-Galerkin method. The solution of the differential (or integrodifferential) problem is searched as a series of wavelets and it is determined (up to a given approximation) when its wavelet coefficients are computed (from an equivalent ordinary differential problem).

Wavelets are special functions $\psi_k^n(x)$ which depend on two parameters. n is the scale (refinement, compression, or dilation) parameter and k is the localization (translation) parameter. Due to the multiscale approach the approximation by wavelets directly refers to a given scale, so that in each scale the detail coefficients β_k^n describe "local" oscillation. Wavelets seems to be more

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expedient for studying the evolution of phenomena which are localized (in time or in frequency) and each scale of approximation tell us about the contribution of that level to the whole event. Moreover, since the solution is represented as a combination of uncorrelated functions they can offer a suitable tool for the analysis of nonlinear problems (see, e.g., [16–18]).

The Petrov-Galerkin wavelet method shortly consists in a transformation of a PDE (integro, integrodifferential equation) into an equivalent ordinary differential problem for the wavelet coefficient. This process is mainly based on the orthogonality of the basis functions and on the computation of the inner product of the basis functions with their derivatives. For linear PDE, this leads us to the computation of the integrals,

$$\gamma^{(\ell)nm}_{\ \ kh} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \frac{\mathrm{d}^\ell \psi_k^n}{\mathrm{d}x^\ell} \overline{\psi_h^m} \, \mathrm{d}x, \tag{1}$$

also called connection coefficients (or refinable integrals), which are the projection of the various order derivatives of the basis functions along with the orthonormal wavelet basis. In particular, for the harmonic wavelet bases [19], which are complex functions (with compact support in the frequency domain), a finite formula for their computation will be given in the following (for any value of ℓ). Harmonic wavelets are complex functions and band-limited in the Fourier domain [19–22], so that they can be used to analyze frequency changes as well as oscillations in a small range time interval. However, many basic problems, in the (wavelet) Petrov-Galerkin method, are still open both for an easy treatment of the boundary conditions and for the computation of the connection coefficients [20,23–26]. Moreover, in nonlinear problems some more general coefficients, with hard numerical problems, appear (see Section 4, where they will be briefly sketched). Due to their difficult computation, the connection coefficients were obtained only in some special cases: either up to a finite given order of the derivatives [25] or with approximate formulas (see e.g., [20]). In general, the computation of the connection coefficients seems to be a difficult task for two reasons. First, the most known (and used) wavelets are not functionally defined by a finite formula. Second, even in the presence of a simple formula defining the wavelet family, such as Daubechies family [27], there does not exist a simple expression for the corresponding connection coefficients [20]. Moreover, except for the first and second derivative, there would not exist explicit formulas for higher derivatives (for the connection coefficients of periodized harmonic wavelets see also [22,28]). In order to have some connection coefficients which can be defined for any order derivative a necessary condition would be to start from some wavelets which are C^∞ functions. This is not enough because integrals (1) could, even in this case, not be easily calculated.

In some previous papers [21,22], it has been shown that it is expedient to make use of complex wavelets with finite support and analytically defined. They can easily represent not only periodic functions but also functions with compact support, i.e., which are defined on a small range interval. Complex wavelets were used not only to solve some PDE problems [22,25] but also in order to extract the wave characteristic [29] as it is usually done in complex demodulation, where the amplitude and phase are decomposed in fundamental signals (at different levels of approximation). Complex harmonic wavelets [19] were applied to the solution of evolution problems [20–22,25,28]. In particular, in [20], the authors give the solution of some nonlinear Burgers equation by using a restricted set of connection coefficients for harmonic wavelets.

The paper is organized as follows. In Section 2, some preliminary definitions about harmonic (complex wavelets) are given. The connection coefficients (1) are computed in Section 3 and a simple finite formula (27) is proven. The same section describes some recursive formulas and properties of the connections coefficients. Section 4 addresses the problem of the computation of connection coefficients for nonlinear problems (only for quadratic nonlinearities). Section 5 briefly summarizes the wavelet Petrov-Galerkin method which will be further investigated in a forthcoming paper.

2. HARMONIC WAVELETS

The harmonic scaling function [19] (see Figure 1),

$$\varphi(x) \stackrel{\text{def}}{=} \frac{e^{2\pi i x} - 1}{2\pi i x} \tag{2}$$

is a complex function whose real and imaginary part are

$$\Re\left(\varphi(x)\right) = \frac{\left(e^{2\pi i x} - e^{-2\pi i x}\right)}{4\pi i x},$$

and

$$\Im\left(\varphi(x)\right) = \frac{-e^{-2\pi i x}\left(e^{2\pi i x} - 1\right)^2}{4\pi x} = \frac{\left(1 - e^{-2\pi i x}\right)\left(1 + e^{2\pi i x}\right)}{4\pi x},$$

and in trigonometric form,

$$\begin{aligned} \Re\left(\varphi(x)\right) &= \frac{\sin 2\pi x}{2\pi x}, \\ \Im\left(\varphi(x)\right) &= \frac{\sin^2 \pi x}{\pi x}. \end{aligned} \tag{3}$$

Since

$$e^{\pi i n} = \begin{cases} 1, & n = 2k, \qquad k \in \mathbb{Z}, \\ -1, & n = 2k+1, \qquad k \in \mathbb{Z}, \end{cases} \tag{4}$$

it is, in particular,

$$\varphi(n) = 0, \qquad n \in \mathbb{Z}. \tag{5}$$

The harmonic scaling function (2) was defined [19] in a such way that its Fourier transform,

$$\hat{\varphi}\left(\omega\right) = \widehat{\varphi\left(x\right)} \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi\left(x\right) e^{-i\omega x}\, dx,$$

results with compact support in the frequency domain (i.e., with bounded frequency),

$$\hat{\varphi}\left(\omega\right) = \frac{1}{2\pi}\chi\left(2\pi + \omega\right). \tag{6}$$

The characteristic (or box) function $\chi(\omega)$, defined (in this paper) as

$$\chi(\omega) \stackrel{\text{def}}{=} \begin{cases} 1, & 2\pi \le \omega < 4\pi, \\ 0, & \text{elsewhere}, \end{cases} \tag{7}$$

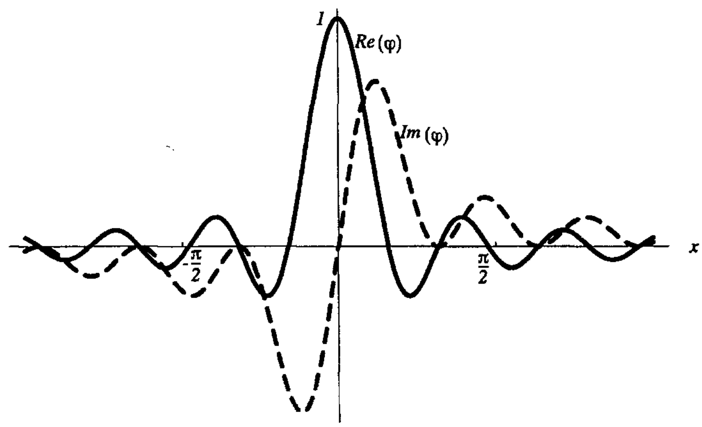

Figure 1. Real (thick line) and imaginary (dashed line) part of the harmonic scaling function $\varphi(x)$.

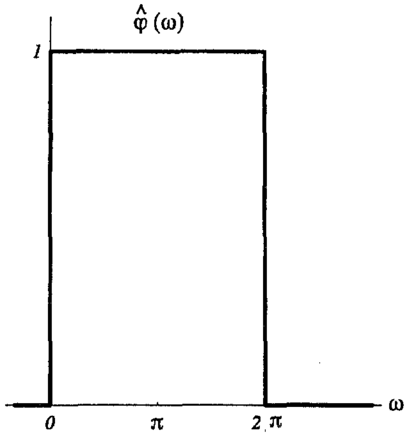

Figure 2. The Fourier transform $\hat{\varphi}(\omega)$ of the scaling function $\varphi(x)$.

is a very well localized function in the frequency domain (Figure 2), despite its slow decay in the space variable. It is easy to check that from (6) by the inverse Fourier transform follows (2),

$$2\pi \int_{-\infty}^{\infty} \frac{1}{2\pi} \chi(2\pi + \omega) e^{i\omega x} \, d\omega = \int_{0}^{2\pi} e^{i\omega x} \, d\omega = \varphi(x).$$

Starting from the scaling function it is possible to define a filter and to derive a corresponding multiresolution analysis based on the harmonic wavelet function (see, e.g., [19]). The harmonic wavelet is the complex valued function [19,25],

$$\psi(x) \stackrel{\text{def}}{=} \frac{e^{4\pi i x} - e^{2\pi i x}}{2\pi i x} = e^{2\pi i x} \varphi(x). \tag{8}$$

The real and imaginary part of the wavelet (8) can be easily derived,

$$\Re\left(\psi(x)\right) = \frac{\left(e^{4\pi i x} - e^{2\pi i x} + e^{-2\pi i x} - e^{-4\pi i x}\right)}{4\pi i x}$$

and

$$\Im\left(\psi(x)\right) = \frac{\left(-e^{4\pi i x} + e^{2\pi i x} + e^{-2\pi i x} - e^{-4\pi i x}\right)}{4\pi x},$$

and in trigonometric form,

$$\begin{aligned} \Re\left(\psi(x)\right) &= \frac{\sin 4\pi x}{2\pi x} - \frac{\sin 2\pi x}{2\pi x}, \\ \Im\left(\varphi(x)\right) &= -\frac{\cos 4\pi x}{2\pi x} + \frac{\cos 2\pi x}{2\pi x}. \end{aligned}$$

In particular, according to (4), (5), (8), it is

$$|\psi(x)| = |\varphi(x)| = \left|\frac{\sin \pi x}{\pi x}\right|, \quad \psi(n) = 0, \qquad n \in \mathbb{Z}. \tag{9}$$

The dilated and translated instances of (8) are (see e.g., [21,22,29])

$$\psi_k^n(x) \stackrel{\text{def}}{=} 2^{n/2} \frac{e^{4\pi i (2^n x - k)} - e^{2\pi i (2^n x - k)}}{2\pi i (2^n x - k)}, \tag{10}$$

with $n, k \in \mathbb{Z}$.

Analogously to (9), for each function of the wavelet family (10), it is

$$|\psi_k^n(x)| = \left|\frac{\sin\pi\,(2^n x - k)}{\pi\,(2^n x - k)}\right|,$$

so that $\lim_{n,k,x\to\infty} |\psi_k^n(x)| = 0$.

The Fourier transform of (10), (see, e.g., [19]) are the band-limited functions,

$$\hat{\psi}_k^n(\omega) = \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n} \chi(\omega/2^n), \tag{11}$$

being, in particular,

$$\hat{\psi}(\omega) = \frac{1}{2\pi}\chi(\omega). \tag{12}$$

In the derivation of equation (11), we have taken into account that for the properties of Fourier transform, if $\hat{f}(\omega)$ is the Fourier transform of $f(x)$, then

$$\widehat{f(ax \pm b)} = \frac{1}{a} e^{\pm i\omega b/a} \hat{f}(\omega/a). \tag{13}$$

Harmonic wavelets are orthonormal functions. In fact, from the definition of the inner (or scalar or dot) product, of two functions $f(x)$, $g(x)$, and taking into account the Parseval equality,

$$\langle f, g\rangle \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} f(x)\,\overline{g(x)}\,\mathrm{d}x = 2\pi \int_{-\infty}^{\infty} \hat{f}(\omega)\,\overline{\hat{g}(\omega)}\,\mathrm{d}\omega = 2\pi\left\langle \hat{f}, \hat{g}\right\rangle, \tag{14}$$

where the bar stands for the complex conjugate. It can be readily shown that harmonic wavelets are orthonormal functions, in the sense that

$$\langle \psi_k^n(x), \psi_h^m(x)\rangle = \delta^{nm}\delta_{hk}, \tag{15}$$

where δ^{nm} (δ_{hk}) is the Kronecker symbol. In fact, it is

$$\begin{aligned}\langle \psi_k^n(x), \psi_h^m(x)\rangle &= 2\pi \int_{-\infty}^{\infty} \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n} \chi(\omega/2^n) \frac{2^{-m/2}}{2\pi} e^{i\omega h/2^m} \chi(\omega/2^m)\,\mathrm{d}\omega \\ &= \frac{2^{-(n+m)/2}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega k/2^n} \chi(\omega/2^n) e^{i\omega h/2^m} \chi(\omega/2^m)\,\mathrm{d}\omega\end{aligned}$$

which is zero for $n \neq m$. (For an alternative proof, see also [19].) For $n = m$, it is

$$\langle \psi_k^n(x), \psi_h^n(x)\rangle = \frac{2^{-n}}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(h-k)/2^n} \chi(\omega/2^n)\,\mathrm{d}\omega.$$

Moreover, according to (7), by the change of variable $\xi = \omega/2^n$,

$$\langle \psi_k^n(x), \psi_h^n(x)\rangle = \frac{1}{2\pi} \int_{2\pi}^{4\pi} e^{-i(h-k)\xi}\,\mathrm{d}\xi.$$

For $h = k$ (and $n = m$), trivially, one has

$$\langle \psi_k^n(x), \psi_k^n(x)\rangle = 1,$$

while for $h \neq k$, it is

$$\int_{2\pi}^{4\pi} e^{-i(h-k)\xi}\,\mathrm{d}\xi = \frac{i}{(h-k)}\left(e^{-4i\pi(h-k)} - e^{-2i\pi(h-k)}\right).$$

and since, according to (4),

$$e^{\pm 4i\pi(h-k)} = e^{\pm 2i\pi(h-k)} = 1 \qquad (h-k \in \mathbb{Z}), \tag{16}$$

the proof easily follows.

Direct computations also show that

$$\left\langle \psi_k^n(x), \overline{\psi_h^m}(x)\right\rangle = 0.$$

It can be shown by technical calculations that the harmonic scaling function and the harmonic wavelets fulfill the multiresolution conditions,

$$\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x = 1, \qquad \int_{-\infty}^{\infty} \psi_k^n(x)\,\mathrm{d}x = 0.$$

Indeed, according to (6)–(14), one has

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x &= \langle 1, \varphi(x)\rangle = 2\pi\left\langle \hat{1}, \hat{\varphi}(\omega)\right\rangle \\ &= 2\pi \int_{-\infty}^{\infty} \delta(\omega)\frac{1}{2\pi}\chi(2\pi+\omega)\,\mathrm{d}\omega \\ &= \int_0^{2\pi} \delta(\omega)\,\mathrm{d}\omega = 1,\end{aligned}$$

where $\delta(\omega)$ is the Dirac delta function.

Analogously, taking into account (11)–(14),

$$\begin{aligned}\int_{-\infty}^{\infty} \psi_k^n(x)\,\mathrm{d}x &= \langle 1, \psi_k^n(x)\rangle = 2\pi\left\langle \hat{1}, \hat{\psi}_k^n(\omega)\right\rangle \\ &= 2\pi \int_{-\infty}^{\infty} \delta(\omega)\frac{2^{-n/2}}{2\pi}e^{-i\omega k/2^n}\chi(\omega/2^n)\,\mathrm{d}\omega \\ &= \int_{2^{n+1}\pi}^{2^{n+2}\pi} \delta(\omega)e^{-i\omega k/2^n}\,\mathrm{d}\omega = 0.\end{aligned}$$

3. CONNECTION COEFFICIENTS FOR HARMONIC WAVELETS

Equation (14) describes the basic structure of the functional space defined on the basis functions (10). The investigation of the differential properties of the basis leads us to the computation of their derivatives. Moreover, in the application of the Galerkin-Petrov method, it is assumed that a certain unknown function (with its derivatives) can be expressed in terms of a basis (and its derivatives). For this reason, as a first step, towards the application of the wavelet Galerkin-Petrov method to PDE, we need the computation of the derivatives of the wavelet basis. Indeed, it is enough the calculation of the connection coefficients which are the components (with respect to the wavelet basis) of the derivatives.

The first- and second-order derivatives of harmonic wavelets are

$$\frac{d\psi_k^n(x)}{dx} = \frac{2^{-1+3n/2}\left[i - 4k\pi + 2^{2+n}\pi x + e^{2i\pi(k-2^n x)}\left(-i + 2k\pi - 2^{1+n}\pi x\right)\right]}{e^{4i\pi(k-2^n x)}\pi(k-2^n x)^2},$$

and, respectively, (Figure 3)

$$\begin{aligned}\frac{d^2\psi_k^n(x)}{dx^2} &= \frac{2^{n/2}4^n\left[i - 8ik^2\pi^2 + 2^{2+n}\pi x - i2^{3+2n}\pi^2x^2 + 4ik\pi\left(i + 2^{2+n}\pi x\right)\right]}{e^{4i\pi(k-2^n x)}\pi(k-2^n x)^3} \\ &\quad + \frac{2^{n/2}4^n\left[\left(2ik^2\pi^2 + 2k\pi\left(1 - i2^{1+n}\pi x\right) + i\left(-1 + i2^{1+n}\pi x + 2^{1+2n}\pi^2x^2\right)\right)\right]}{e^{2i\pi(k-2^n x)}\pi(k-2^n x)^3}.\end{aligned}$$

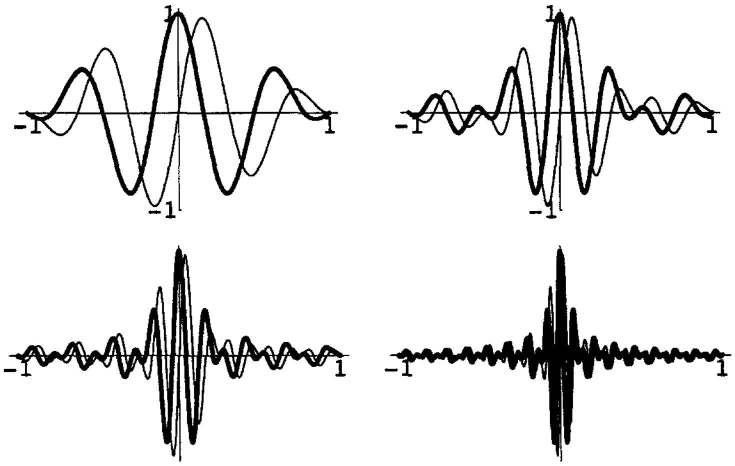

Figure 3. Real (thick line) and imaginary (thin line) part of the harmonic wavelets $\psi_0^0(x)$, $\psi_0^1(x)$, (first row) and $\psi_0^2(x)$, $\psi_0^3(x)$ (second row).

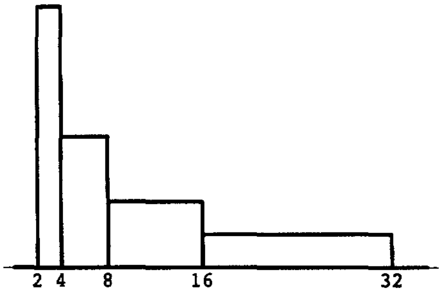

Figure 4. The Fourier transform of the harmonic wavelets $\psi_0^0(x)$, $\psi_0^1(x)$, and $\psi_0^2(x)$, $\psi_0^3(x)$.

They are infinitely differentiable functions, $\psi_k^n(x) \in C^\infty$, in fact, it can be easily checked that in correspondence of $x = k/2^n$,

$$\lim_{x \to k/2^n} \frac{d^p \psi_k^n(x)}{dx^p} \propto 2^{(p+2)n/2} i^n \pi^n < \infty,$$

we have a vanishing singularity. However, in the x-domain the computation of higher-order derivatives implies cumbersome formulas. Therefore, it is more expedient to compute the derivatives in the Fourier domain. In the Fourier domain, the n-order derivative of the wavelet basis is

$$\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \psi_k^n(x)} = (i\omega)^\ell \hat{\psi}_k^n(\omega), \tag{17}$$

and, according to (11),

$$\widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \psi_k^n(x)} = (i\omega)^\ell \frac{2^{-n/2}}{2\pi} e^{-i\omega k/2^n} \chi(\omega/2^n). \tag{18}$$

3.1. Recursive Preliminary Formulas

In order to define the any order differential properties of harmonic wavelets, in the following, we will meet the integrals $\int_{2\pi}^{4\pi} \xi^\ell e^{im\xi}\,\mathrm{d}\xi$. Usually, they might be expressed as

$$\int_{2\pi}^{4\pi} \xi^\ell e^{im\xi}\,\mathrm{d}\xi = \frac{1}{(-im)^{\ell+1}} \left[\Gamma(\ell+1,\xi)\right]_{-4im\pi}^{-2im\pi},$$

being $\Gamma(\ell,\xi) \stackrel{\text{def}}{=} \int_\xi^\infty t^{\ell-1} e^{-t}\,\mathrm{d}t$, the Euler Gamma function, and $[F(\xi)]_{\xi_0}^{\xi_1} \stackrel{\text{def}}{=} F(\xi_1) - F(\xi_0)$. However, it is possible to give them an explicit recursive formula as follows.

LEMMA 1. *For a given $m \in \mathbb{Z}$, $\ell \in \mathbb{N} \cup \{0\}$, and $\xi \in [a,b] \subset \mathbb{R}$, it is*

$$\int_a^b \xi^\ell e^{im\xi}\,\mathrm{d}\xi = \begin{cases} -\dfrac{\ell}{im}\int_a^b \xi^{\ell-1}e^{im\xi}\,\mathrm{d}\xi + \left[\dfrac{1}{im}\xi^\ell e^{im\xi}\right]_a^b, & \ell \geq 1, \\ \left[\dfrac{1}{im}e^{im\xi}\right]_a^b, & \ell = 0. \end{cases}$$

PROOF. It can be easily obtained by partial integration. ∎

Moreover, if we define $I^\ell \stackrel{\text{def}}{=} \int_a^b \xi^\ell e^{im\xi}\,\mathrm{d}\xi$, there results

$$\begin{aligned} I^0 &= \left[\frac{1}{im}e^{im\xi}\right]_a^b, \\ I^1 &= -\frac{1}{im}\left[\frac{1}{im}e^{im\xi}\right]_a^b + \left[\frac{1}{im}\xi e^{im\xi}\right]_a^b, \\ I^2 &= -\frac{2}{im}\left[-\frac{1}{im}\left(\frac{1}{im}e^{im\xi}\right) + \left(\frac{1}{im}\xi e^{im\xi}\right)\right]_a^b + \left[\frac{1}{im}(\xi e^{im\xi})^2\right]_a^b, \\ &\vdots \quad \dots, \end{aligned}$$

giving rise to some cumbersome formulas. However, in some cases, they admit some simple recursive expressions. In fact, we have the following lemma.

LEMMA 2. *For a given $m \in \mathbb{Z}/\{0\}$ and $\ell \in \mathbb{N} \cup \{0\}$, it is*

$$\int_{2\pi}^{4\pi} \xi^\ell e^{im\xi}\,\mathrm{d}\xi = \sum_{k=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,(2\pi)^k\,(2^k-1)}{k!\,(im)^{\ell-k+1}}, \qquad \ell \geq 1, \tag{19}$$

and

$$\int_{2\pi}^{4\pi} \xi^\ell e^{im\xi}\,\mathrm{d}\xi = 0, \qquad \ell = 0.$$

PROOF. According to Lemma 1, it is

$$\begin{aligned} \int_{2\pi}^{4\pi} \xi^\ell e^{im\xi}\,\mathrm{d}\xi &= -\frac{\ell}{im}\int_{2\pi}^{4\pi}\xi^{\ell-1}e^{im\xi}\,\mathrm{d}\xi + \left[\frac{1}{im}\xi^\ell e^{im\xi}\right]_{2\pi}^{4\pi}, & \ell \geq 1, \\ \int_{2\pi}^{4\pi} \xi^\ell e^{im\xi}\,\mathrm{d}\xi &= 0, & \ell = 0, \end{aligned}$$

i.e., if we define $I^\ell \stackrel{\text{def}}{=} \int_{2\pi}^{4\pi} \xi^\ell e^{im\xi}\,\mathrm{d}\xi$, and we take into account (4), we have the recursive formula,

$$I^0 = 0, \quad I^\ell = \frac{-\ell}{im}I^{\ell-1} + \left[\frac{1}{im}\xi^\ell e^{im\xi}\right]_{2\pi}^{4\pi}, \qquad \ell \geq 1, \tag{20}$$

and explicitly, there results

$$\begin{aligned} I^0 &= 0, \\ I^1 &= \left[\frac{1}{im}\xi\right]_{2\pi}^{4\pi}, \\ I^2 &= -\frac{2}{im}\left[\frac{1}{im}\xi\right]_{2\pi}^{4\pi} + \left[\frac{1}{im}\xi^2\right]_{2\pi}^{4\pi}, \\ I^3 &= -\frac{3}{im}\left\{-\frac{2}{im}\left[\frac{1}{im}\xi\right]_{2\pi}^{4\pi} + \left[\frac{1}{im}\xi^2\right]_{2\pi}^{4\pi}\right\} + \left[\frac{1}{im}\xi^3\right]_{2\pi}^{4\pi}, \\ I^4 &= -\frac{4}{im}\left\{-\frac{3}{im}\left\{-\frac{2}{im}\left[\frac{1}{im}\xi\right]_{2\pi}^{4\pi} + \left[\frac{1}{im}\xi^2\right]_{2\pi}^{4\pi}\right\} + \left[\frac{1}{im}\xi^3\right]_{2\pi}^{4\pi}\right\} + \left[\frac{1}{im}\xi^4\right]_{2\pi}^{4\pi}, \\ &\vdots \quad \dots \end{aligned}$$

So, we can write

$$
\begin{aligned}
I^0 &= 0, \\
I^1 &= \left[\frac{\xi}{im}\right]_{2\pi}^{4\pi}, \\
I^2 &= \left[-2\cdot 1\frac{\xi}{(im)^2} + \frac{\xi^2}{im}\right]_{2\pi}^{4\pi}, \\
I^3 &= \left[3\cdot 2\cdot 1\frac{\xi}{(im)^3} - 3\cdot\frac{\xi^2}{(im)^2} + \frac{\xi^3}{im}\right]_{2\pi}^{4\pi}, \\
I^4 &= \left[-4\cdot 3\cdot 2\cdot 1\frac{\xi}{(im)^4} + 4\cdot 3\frac{\xi^2}{(im)^3} - 4\frac{\xi^3}{(im)^2} + \frac{\xi^4}{im}\right]_{2\pi}^{4\pi}, \\
\vdots\ & \ \dots, \\
I^\ell &= \left[\sum_{k=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,\xi^k}{k!\,(im)^{\ell-k+1}}\right]_{2\pi}^{4\pi}.
\end{aligned}
\tag{21}
$$

It is also

$$
I^\ell = \left[\frac{1}{(im)^{\ell+1}}\sum_{k=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,(\xi/(im))^k}{k!}\right]_{2\pi}^{4\pi}.
$$

By explicit computation of the last one, there follows,

$$
\begin{aligned}
I^\ell &= \left[\sum_{k=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,(4\pi)^k}{k!\,(im)^{\ell-k+1}}\right] - \left[\sum_{k=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,(2\pi)^k}{k!\,(im)^{\ell-k+1}}\right] \\
&= \sum_{k=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,\left[(4\pi)^k - (2\pi)^k\right]}{k!\,(im)^{\ell-k+1}} \\
&= \sum_{k=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,(2\pi)^k\,(2^k-1)}{k!\,(im)^{\ell-k+1}}.
\end{aligned}
$$

∎

There immediately follows the explicit recursive formula.

THEOREM 1. *For a given $m \in \mathbb{Z}/\{0\}$ and $\ell \in \mathbb{N}$, it is*

$$
\int_{2\pi}^{4\pi}\xi^{\ell+1}e^{im\xi}\,\mathrm{d}\xi = -\frac{\ell+1}{im}\int_{2\pi}^{4\pi}\xi^{\ell}e^{im\xi}\,\mathrm{d}\xi + \frac{(2\pi)^{\ell+1}\,(2^{\ell+1}-1)}{im}, \qquad \ell \ge 1. \tag{22}
$$

PROOF. It is a direct consequence of equations (4),(20). ∎

The same integral (19) can be expressed in another way (see e.g., [22,28]). In particular, when the upper bound limit of the integral is ∞, then the integrals can be expressed in term of Gamma functions.

THEOREM 2. *For a given $m \in \mathbb{Z}/\{0\}$ and $\ell \in \mathbb{N}$, the following holds true,*

$$
\begin{aligned}
\int_{2\pi}^{4\pi}\xi^{\ell}e^{im\xi}\,\mathrm{d}\xi &= \sum_{k=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,(2\pi)^k\,(2^k-1)}{k!\,(im)^{\ell-k+1}} \\
&= \left[\sum_{k=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,\xi^k}{k!\,(im)^{\ell-k+1}}\right]_{2\pi}^{4\pi} \\
&= -\left[e^{im\xi}\sum_{j=0}^{\ell}a_j^{(\ell)}\,\xi^{\ell-j}\left(\frac{i}{m}\right)^{j+1}\right]_{2\pi}^{4\pi}, \qquad \ell \ge 1,
\end{aligned}
$$

with

$$a_0^{(0)} = 1, \qquad a_j^{(\ell)} = \begin{cases} 1, & j = \ell, \\ \ell a_j^{(\ell-1)}, & j = 0, \dots, \ell - 1, \end{cases} \qquad \ell \geq 1. \tag{23}$$

PROOF. It is enough to show that

$$\left[\sum_{k=1}^{\ell} (-1)^{\ell-k} \frac{\ell!\, \xi^k}{k!\, (im)^{\ell-k+1}}\right]_{2\pi}^{4\pi} = -\left[e^{im\xi} \sum_{j=0}^{\ell} a_j^{(\ell)}\, \xi^{\ell-j} \left(\frac{i}{m}\right)^{j+1}\right]_{2\pi}^{4\pi}.$$

If we define $I^\ell \stackrel{\text{def}}{=} \int_{2\pi}^{4\pi} \xi^\ell e^{im\xi}\, d\xi$, by explicit computation of the left-hand side, we have equations (21), while the right-hand side gives

$$\begin{aligned}
I^1 &= -\left[e^{im\xi}\left(\frac{1}{m^2} - \frac{i\xi}{m}\right)\right]_{2\pi}^{4\pi}, \\
I^2 &= -\left[e^{im\xi}\left(\frac{2i}{m^3} + \frac{2\xi}{m^2} - \frac{i\xi^2}{m}\right)\right]_{2\pi}^{4\pi}, \\
I^3 &= -\left[e^{im\xi}\left(-\frac{6}{m^4} + \frac{6i\xi}{m^3} + \frac{3\xi^2}{m^2} - \frac{i\xi^3}{m}\right)\right]_{2\pi}^{4\pi}, \\
I^4 &= -\left[e^{im\xi}\left(\frac{-24\,i}{m^5} - \frac{24\,\xi}{m^4} + \frac{12i\xi^2}{m^3} + \frac{4\,\xi^3}{m^2} - \frac{i\,\xi^4}{m}\right)\right]_{2\pi}^{4\pi}, \\
&\ \vdots
\end{aligned} \tag{24}$$

Since $[e^{im\xi}\text{const}]_{2\pi}^{4\pi} = 0$ and $[e^{im\xi} f(\xi)]_{2\pi}^{4\pi} = [f(\xi)]_{2\pi}^{4\pi}$, there follows,

$$\begin{aligned}
I^1 &= -\left[-\frac{i\xi}{m}\right]_{2\pi}^{4\pi}, \\
I^2 &= -\left[\frac{2\xi}{m^2} - \frac{i\xi^2}{m}\right]_{2\pi}^{4\pi}, \\
I^3 &= -\left[\frac{6i\xi}{m^3} + \frac{3\xi^2}{m^2} - \frac{i\xi^3}{m}\right]_{2\pi}^{4\pi}, \\
I^4 &= -\left[-\frac{24\,\xi}{m^4} + \frac{12\,i\,\xi^2}{m^3} + \frac{4\,\xi^3}{m^2} - \frac{i\,\xi^4}{m}\right]_{2\pi}^{4\pi},
\end{aligned} \tag{25}$$

which coincide with the left-hand side (21). ∎

3.2. Connection Coefficients

The any order connection coefficients of the harmonic wavelets, are defined as

$$\gamma^{(\ell)}{}_{kh}^{nm} \stackrel{\text{def}}{=} \left\langle \frac{d^\ell}{dx^\ell} \psi_k^n(x), \psi_h^m(x) \right\rangle. \tag{26}$$

They can be easily computed by the following theorem (for the first- and second-order connection coefficients of *periodic* harmonic wavelets, see also [21,22,25,28]). Taking into account equations (17),(18), they are given by the following theorem.

THEOREM 3. *The connection coefficients (26) of the harmonic wavelets (10) are given by*

$$\gamma^{(\ell)}{}_{kh}^{nm} = \frac{i^\ell 2^{n\ell}}{2\pi} \left[\delta_{kh} \frac{(2\pi)^{\ell+1}}{\ell+1} \left(2^{\ell+1} - 1\right) - (1 - \delta_{kh})^{\ell+1} \sum_{j=1}^{\ell} (-1)^{\ell-j} \frac{\ell!\, (2\pi)^j\, (2^j - 1)}{j!\, [i(h-k)]^{\ell-j+1}}\right] \delta^{nm}, \tag{27}$$

for $\ell \geq 1$, and

$$\gamma^{(\ell)}{}^{nm}_{kh} = \delta_{kh}\delta^{nm},$$

when $\ell = 0$.

PROOF. From their definition (26), taking into account equations (14)–(18), it is

$$\begin{aligned}\left\langle \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\psi^n_k(x), \psi^m_h(x)\right\rangle &= 2\pi \left\langle \widehat{\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\psi^n_k(x)}, \widehat{\psi^m_h(x)}\right\rangle \\ &= 2\pi \left\langle (i\omega)^\ell \hat{\psi}^n_k(\omega), \hat{\psi}^m_h(\omega)\right\rangle \\ &= 2\pi \int_{-\infty}^{\infty} \frac{2^{-n/2}}{2\pi}(i\omega)^\ell e^{-i\omega k/2^n}\chi(\omega/2^n)\frac{2^{-m/2}}{2\pi}e^{i\omega h/2^m}\chi(\omega/2^m)\,\mathrm{d}\omega,\end{aligned}$$

that is,

$$\gamma^{(\ell)}{}^{nm}_{kh} = \frac{2^{-(n+m)/2}}{2\pi}\int_{-\infty}^{\infty}(i\omega)^\ell e^{-i\omega k/2^n}\chi(\omega/2^n)e^{i\omega h/2^m}\chi(\omega/2^m)\,\mathrm{d}\omega,$$

which is 0 when $m \neq n$.

When $m = n$ we have, with the change of variable $\omega/2^n = \xi$,

$$\gamma^{(\ell)}{}^{nn}_{kh} = \frac{i^\ell 2^{n\ell}}{2\pi}\int_{2\pi}^{4\pi}\xi^\ell e^{i\xi(h-k)}\,\mathrm{d}\xi. \tag{28}$$

So that, when $k = h$, we obtain

$$\gamma^{(\ell)}{}^{nn}_{kk} = \frac{i^\ell 2^{n\ell}}{2\pi(\ell+1)}\left[\xi^{\ell+1}\right]_{2\pi}^{4\pi} = \frac{i^\ell 2^{n\ell}}{2\pi}\left[\frac{(2\pi)^{\ell+1}}{\ell+1}\left(2^{\ell+1}-1\right)\right]. \tag{29}$$

When $k \neq h$, by using (19) of Lemma 2, we have

$$\gamma^{(\ell)}{}^{nn}_{kh} = \frac{i^\ell 2^{n\ell}}{2\pi}\sum_{j=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,(2\pi)^j\left(2^j-1\right)}{j!\,\left[i(h-k)\right]^{\ell-j+1}}, \qquad \ell \geq 1, \tag{30}$$

and since we want $\gamma^{(\ell)}{}^{nn}_{kh}$ to be zero when $k = h$, with a product with

$$\left(1-\delta_{kh}\right)^{\ell+1},$$

we get

$$\gamma^{(\ell)}{}^{nn}_{kh} = \frac{i^\ell 2^{n\ell}}{2\pi}\left(1-\delta_{kh}\right)^{\ell+1}\sum_{j=1}^{\ell}(-1)^{\ell-k}\frac{\ell!\,(2\pi)^j\left(2^j-1\right)}{j!\,\left[i(h-k)\right]^{\ell-j+1}}, \qquad \ell \geq 1. \tag{31}$$

The last part of the proof ($\ell = 0$), easily follows from the definitions of harmonic wavelets and inner product. ∎

Thus, by applying equation (27), we obtain $k \neq h$,

$$\begin{aligned}\gamma^{(1)}{}^{nn}_{kh} &= \frac{i\,2^{n-1}}{\pi}\left[\frac{2\pi i}{(h-k)}\right], \\ \gamma^{(2)}{}^{nn}_{kh} &= \frac{2^{2n-1}}{\pi}\left[\frac{4\pi}{(h-k)^2} - \frac{12\,\pi^2 i}{(h-k)}\right], \\ \gamma^{(3)}{}^{nn}_{kh} &= \frac{i2^{3n-1}}{\pi}\left[\frac{12\pi i}{(h-k)^3} + \frac{36\pi^2}{(h-k)^2} - \frac{56\pi^3 i}{(h-k)}\right],\end{aligned}$$

and for $k = h$,

$$\gamma^{(1)nn}_{\ \ kk} = 3i\pi 2^n,$$
$$\gamma^{(2)nn}_{\ \ kk} = -\frac{7\pi^2 2^{2+2n}}{3},$$
$$\gamma^{(3)nn}_{\ \ kk} = (-15i)\, 2^{1+3n}\pi^3.$$

So that, in particular, assuming $k = 0, \dots 2^n - 1$, $h = 0, \dots 2^m - 1$ from (27), we have the following.

- For the first-order ($\ell = 1$) coefficients $\gamma^{(1)nm}_{\ \ kh}$, at the lower scales $0 \le n = m \le 3$,

$$\gamma^{(1)00}_{\ \ 00} = 3\pi i,$$

$$\gamma^{(1)11}_{\ \ kh} = \begin{pmatrix} 6i\pi & -2 \\ 2 & 6i\pi \end{pmatrix},$$

$$\gamma^{(1)22}_{\ \ kh} = \begin{pmatrix} 12i\pi & -4 & -2 & -\frac{4}{3} \\ 4 & 12i\pi & -4 & -2 \\ 2 & 4 & 12i\pi & -4 \\ \frac{4}{3} & 2 & 4 & 12i\pi \end{pmatrix},$$

$$\gamma^{(1)33}_{\ \ kh} = \begin{pmatrix} 24i\pi & -8 & -4 & -\frac{8}{3} & -2 & -\frac{8}{5} & -\frac{4}{3} & -\frac{8}{7} \\ 8 & 24i\pi & -8 & -4 & -\frac{8}{3} & -2 & -\frac{8}{5} & -\frac{4}{3} \\ 4 & 8 & 24i\pi & -8 & -4 & -\frac{8}{3} & -2 & -\frac{8}{5} \\ \frac{8}{3} & 4 & 8 & 24i\pi & -8 & -4 & -\frac{8}{3} & -2 \\ 2 & \frac{8}{3} & 4 & 8 & 24i\pi & -8 & -4 & -\frac{8}{3} \\ \frac{8}{5} & 2 & \frac{8}{3} & 4 & 8 & 24i\pi & -8 & -4 \\ \frac{4}{3} & \frac{8}{5} & 2 & \frac{8}{3} & 4 & 8 & 24i\pi & -8 \\ \frac{8}{7} & \frac{4}{3} & \frac{8}{5} & 2 & \frac{8}{3} & 4 & 8 & 24i\pi \end{pmatrix}.$$

- For the second-order ($\ell = 2$) coefficients $\gamma^{(2)nm}_{\ \ kh}$, at the lower scales $0 \le n = m \le 2$,

$$\gamma^{(2)00}_{\ \ 00} = -\frac{28\pi^2}{3},$$

$$\gamma^{(2)11}_{\ \ kh} = \begin{pmatrix} -\frac{112}{3}\pi^2 & -8 - 24i\pi \\ -8 + 24i\pi & -\frac{112}{3}\pi^2 \end{pmatrix},$$

$$\gamma^{(2)22}_{\ \ kh} = \begin{pmatrix} -\frac{448}{3}\pi^2 & -32 - 96i\pi & -8 - 48i\pi & -\frac{32}{9} - 32i\pi \\ -32 + 96i\pi & -\frac{448}{3}\pi^2 & -32 - 96i\pi & -8 - (48i\pi \\ -8 + 48i\pi & -32 + 96i\pi & -\frac{448}{3}\pi^2 & -32 - 96i\pi \\ -\frac{32}{9} + 32i\pi & -8 + 48i\pi & -32 + 96i\pi & -\frac{448}{3}\pi^2 \end{pmatrix}.$$

- For the third order ($\ell = 3$) coefficients $\gamma^{(3)}{}_{kh}^{nm}$, at the lower scales $0 \le n = m \le 2$,

$$\gamma^{(3)}{}_{00}^{00} = -30i\pi^3,$$

$$\gamma^{(3)}{}_{kh}^{11} = \begin{pmatrix} -240i\pi^3 & -48 - 144i\pi + 224\pi^2 \\ 48 - 144i\pi - 224\pi^2 & -240i\pi^3 \end{pmatrix},$$

$$\gamma^{(3)}{}_{kh}^{22} =$$

$$\begin{pmatrix} -1920i\pi^3 & 128\left(-3 - 9i\pi + 14\pi^2\right) & 16\left(-3 - 18i\pi + 56\pi^2\right) & \frac{128}{9}\left(-1 - 9i\pi + 42\pi^2\right) \\ -128\left(-3 + 9i\pi + 14\pi^2\right) & -1920i\pi^3 & 128\left(-3 - 9i\pi + 14\pi^2\right) & 16\left(-3 - 18i\pi + 56\pi^2\right) \\ -16\left(-3 + 18i\pi + 56\pi^2\right) & -128\left(-3 + 9i\pi + 14\pi^2\right) & -1920i\pi^3 & 128\left(-3 - 9i\pi + 14\pi^2\right) \\ -\frac{128}{9}\left(-1 + 9i\pi + 42\pi^2\right) & -16\left(-3 + 18i\pi + 56\pi^2\right) & -128\left(-3 + 9i\pi + 14\pi^2\right) & -1920i\pi^3 \end{pmatrix},$$

Concerning the connection coefficients of the conjugate basis $\overline{\psi_k^n}(x)$, it can be easily proven that

$$\overline{\gamma^{(\ell)}{}_{kh}^{nm}} \stackrel{\text{def}}{=} \left\langle \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\overline{\psi_k^n}(x), \overline{\psi_k^n}(x) \right\rangle = -\gamma^{(\ell)}{}_{kh}^{nm}.$$

3.3. Recursive Equations for the Connection Coefficients of Harmonic Wavelets

The connection coefficients (27) of different orders are not independent. In fact, they can be constructed according to the following.

THEOREM 4. *The connection coefficients (27) are recursively given by*

$$\begin{aligned} \gamma^{(\ell+1)}{}_{kh}^{nm} &= \left[\delta_{kh}(i\pi)2^{n+1}\frac{(\ell+1)(2^{\ell+2}-1)}{(\ell+2)(2^{\ell+1}-1)} + (1-\delta_{kh})\frac{-2^n\ell}{(h-k)}\right]\gamma^{(\ell)}{}_{kh}^{nm} \\ &\quad + \frac{i^\ell\pi^{\ell-1}2^{\ell(n+1)+n-1}}{(h-k)}\left(2^\ell - 1\right)\left(1-\delta_{kh}\right)\delta^{nm}, \\ \gamma^{(1)}{}_{kh}^{nm} &= \left[3(i\pi)2^n\delta_{kh} + (1-\delta_{kh})\frac{2^n}{(h-k)}\right]\delta^{nm} \end{aligned} \tag{32}$$

PROOF. Assuming $n = m$, let us first consider the case $h \neq k$. It can be easily seen by a direct computation (but also from the explicit values of $\gamma^{(\ell)}{}_{kh}^{nn}$ in (27)) that the polynomials,

$$\varepsilon_{hk}^{(\ell)}(\xi) \stackrel{\text{def}}{=} -\sum_{j=0}^{\ell} a_j^{(\ell)}\xi^{\ell-j}\left(\frac{i}{h-k}\right)^{j+1},$$

fulfill the recursive equations

$$\varepsilon_{hk}^{(\ell+1)}(\xi) = \frac{i\ell}{(h-k)}\varepsilon_{hk}^{(\ell)}(\xi) - e^{i\xi(h-k)}\frac{i\xi^\ell}{(h-k)},$$

$$\varepsilon_{hk}^{(1)}(\xi) = e^{i\xi(h-k)}\left(\frac{1}{(h-k)^2} - \frac{i\xi}{(h-k)}\right),$$

so that (when $h \neq k$), it is

$$\begin{aligned} \gamma^{(\ell+1)}{}_{kh}^{nn} &= \frac{i^{(\ell+1)}2^{n(\ell+1)}}{2\pi}\left[\varepsilon_{hk}^{(\ell+1)}(\xi)\right]_{2\pi}^{4\pi} \\ &= \frac{i^{(\ell+1)}2^{n(\ell+1)}}{2\pi}\left[\frac{i\ell}{(h-k)}\varepsilon_{hk}^{(\ell)}(\xi) - e^{i\xi(h-k)}\frac{i\xi^\ell}{(h-k)}\right]_{2\pi}^{4\pi} \\ &= \frac{-2^n\ell}{(h-k)}\left[\frac{i^\ell 2^{n\ell}}{2\pi}\varepsilon_{hk}^{(\ell)}(\xi)\right]_{2\pi}^{4\pi} + \frac{i^\ell 2^{n(\ell+1)}}{2\pi}\left[e^{i\xi(h-k)}\frac{\xi^\ell}{(h-k)}\right]_{2\pi}^{4\pi}. \end{aligned}$$

Since $[e^{i\xi(h-k)}\xi^{\ell}/(h-k)]_{2\pi}^{4\pi} = (2\pi)^{\ell}\left(2^{\ell}-1\right)/(h-k)$, and

$$\left[e^{i\xi(h-k)}\left(\frac{1}{(h-k)^2}-\frac{i\xi}{(h-k)}\right)\right]_{2\pi}^{4\pi} = -\frac{2i\pi}{(h-k)},$$

we finally obtain

$$\begin{aligned}\gamma^{(\ell+1)}{}_{kh}^{nn} &= \frac{-2^n\ell}{(h-k)}\gamma^{(\ell)}{}_{kh}^{nn} + \frac{i^{\ell}\pi^{\ell-1}2^{\ell(n+1)+n-1}}{(h-k)}\left(2^{\ell}-1\right),\\ \gamma^{(1)}{}_{kh}^{nn} &= \frac{2^n}{(h-k)}.\end{aligned}$$

Analogously, we can easily derive the recursive formula when $k = h$. From (27), it is

$$\gamma^{(\ell+1)}{}_{kk}^{nn} = \frac{(i\pi)^{\ell+1}2^{(\ell+1)(n+1)}}{\ell+2}\left(2^{\ell+2}-1\right),$$

from where

$$\begin{aligned}\gamma^{(\ell+1)}{}_{kk}^{nn} &= (i\pi)2^{n+1}\frac{(\ell+1)(2^{\ell+2}-1)}{(\ell+2)(2^{\ell+1}-1)}\gamma^{(\ell)}{}_{kk}^{nn},\\ \gamma^{(1)}{}_{kk}^{nn} &= 3(i\pi)2^n.\end{aligned}$$

There follows the explicit recursive formula (32) for any value of the indices. ∎

The connection coefficients (27) as well as the recursive equation (32), can be written in a simple form if we define the following complex function,

$$\gamma_{kh}^{(\ell)}(\xi) \stackrel{\text{def}}{=} \delta_{kh}\frac{\xi^{\ell+1}}{\ell+1} - (1-\delta_{kh})\sum_{j=0}^{\ell} a_j^{(\ell)}\xi^{\ell-j}\left(\frac{i}{h-k}\right)^{j+1}, \tag{33}$$

Then, we have the following theorem.

THEOREM 5. *The connection coefficients (27) are given by*

$$\gamma^{(\ell+1)}{}_{kh}^{nm} = \frac{i^{\ell+1}2^{n(\ell+1)}}{2\pi}\left[\gamma_{kh}^{(\ell+1)}(\xi)\right]_{2\pi}^{4\pi}\delta^{nm}$$

where the function $\gamma_{kh}^{(\ell)}(\xi)$, defined in (33), is recursively given by

$$\begin{aligned}\gamma_{kh}^{(\ell+1)}(\xi) &= \xi(\ell+1)\left(\frac{\delta_{hk}}{\ell+2}+(1-\delta_{hk})\right)\gamma_{kh}^{(\ell)}(\xi) - (1-\delta_{kh})\left(\frac{\ell+1}{h-k}\right)^{\ell+2} \qquad (\ell \geq 1),\\ \gamma_{kh}^{(1)}(\xi) &= \delta_{kh}\frac{\xi^2}{2} - (1-\delta_{kh})\sum_{j=0}^{1}\xi^{1-j}\left(\frac{i}{h-k}\right)^{j+1}.\end{aligned} \tag{34}$$

PROOF. In fact, it is

$$\gamma_{kh}^{(\ell+1)}(\xi) = \delta_{kh}\xi\frac{\ell+1}{\ell+2}\frac{\xi^{\ell+1}}{\ell+1} - (1-\delta_{kh})\sum_{j=0}^{\ell} a_j^{(\ell+1)}\xi^{\ell-j}\xi\left(\frac{i}{h-k}\right)^{j+1} + -(1-\delta_{kh})\,a_{\ell+1}^{(\ell+1)}\left(\frac{\ell+1}{h-k}\right)^{\ell+2}$$

and according to (23)

$$\gamma_{kh}^{(\ell+1)}(\xi) = \delta_{kh}\xi\frac{\ell+1}{\ell+2}\frac{\xi^{\ell+1}}{\ell+1} - (1-\delta_{kh})\sum_{j=0}^{\ell}(\ell+1)a_j^{(\ell)}\xi^{\ell-j}\xi\left(\frac{i}{h-k}\right)^{j+1} + -(1-\delta_{kh})\left(\frac{\ell+1}{h-k}\right)^{\ell+2},$$

from where

$$\gamma_{kh}^{(\ell+1)}(\xi) = \xi(\ell+1)\left(\frac{\delta_{hk}}{\ell+2} + (1-\delta_{hk})\right)\left[\delta_{kh}\frac{\xi^{\ell+1}}{\ell+1} - (1-\delta_{kh})\sum_{j=0}^{\ell} a_j^{(\ell)}\xi^{\ell-j}\xi\left(\frac{i}{h-k}\right)^{j+1}\right]$$
$$+ - (1-\delta_{kh})\left(\frac{\ell+1}{h-k}\right)^{\ell+2}$$

that is,

$$\gamma_{kh}^{(\ell+1)}(\xi) = \xi(\ell+1)\left(\frac{\delta_{hk}}{\ell+2} + (1-\delta_{hk})\right)\gamma_{kh}^{(\ell)}(\xi) - (1-\delta_{kh})\left(\frac{\ell+1}{h-k}\right)^{\ell+2} \qquad (\ell \geq 1)$$

being

$$\gamma_{kh}^{(1)}(\xi) = \delta_{kh}\frac{\xi^2}{2} - (1-\delta_{kh})\sum_{j=0}^{1}\xi^{1-j}\left(\frac{i}{h-k}\right)^{j+1}.$$

∎

4. CONNECTION COEFFICIENTS FOR NONLINEAR PROBLEMS

In order to solve nonlinear problems (with Petrov-Galerkin method) the nonlinear terms of PDE give rise to some more general connection coefficients. If we restrict ourselves to quadratic terms, we might have

$$\begin{aligned}
\Xi_{jkh}^{snm} &\overset{\text{def}}{=} \int_{-\infty}^{\infty} \psi_j^s \psi_k^n \overline{\psi_h^m}\,\mathrm{d}x, \\
\Gamma^{(\ell)}{}_{jkh}^{snm} &\overset{\text{def}}{=} \int_{-\infty}^{\infty} \psi_j^s \frac{\mathrm{d}^\ell \psi_k^n}{\mathrm{d}x^\ell}\overline{\psi_h^m}\,\mathrm{d}x, \\
\Theta^{(\ell,r)}{}_{jkh}^{snm} &\overset{\text{def}}{=} \int_{-\infty}^{\infty} \frac{\mathrm{d}^\ell \psi_j^s}{\mathrm{d}x^\ell}\frac{\mathrm{d}^r \psi_k^n}{\mathrm{d}x^r}\overline{\psi_h^m}\,\mathrm{d}x.
\end{aligned} \tag{35}$$

The first group of coefficients,

$$\Xi(\ell)_{jkh}^{snm} = \left\langle \psi_j^s \psi_k^n, \psi_h^m \right\rangle, \tag{36}$$

according to equations (14)–(18), can be written as

$$\left\langle \psi_j^s(x)\,\psi_k^n(x), \psi_h^m(x)\right\rangle = 2\pi\left\langle \widehat{\psi_j^s(x)\,\psi_k^n}(x), \widehat{\psi_h^m(x)}\right\rangle.$$

For the properties of the Fourier transform, with respect to the convolution product ($\star$), we have

$$\widehat{\psi_j^s(x)\,\psi_k^n}(x) = \widehat{\psi_j^s(x)} \star \widehat{\psi_k^n(x)} = \int_{-\infty}^{\infty} \hat{\psi}_j^s(\omega-\lambda)\hat{\psi}_k^n(\lambda)\,\mathrm{d}\lambda$$

i.e., taking into account (17),

$$\widehat{\psi_j^s(x)\,\psi_k^n}(x) = \int_{-\infty}^{\infty} \hat{\psi}_j^s(\omega-\lambda)\hat{\psi}_k^n(\lambda)\,\mathrm{d}\lambda$$

and, according to (11),

$$\begin{aligned}
\widehat{\psi_j^s(x)\,\psi_k^n}(x) &= \int_{-\infty}^{\infty} \frac{2^{-s/2}}{2\pi}e^{-i(\omega-\lambda)j/2^s}\chi\left((\omega-\lambda)/2^s\right)\frac{2^{-n/2}}{2\pi}e^{-i\lambda k/2^n}\chi\left(\lambda/2^n\right)\,\mathrm{d}\lambda \\
&= \frac{2^{-s/2-n/2}}{4\pi^2}e^{-i\omega j/2^s}\int_{-\infty}^{\infty} e^{i\lambda(j/2^s-k/2^n)}\chi\left((\omega-\lambda)/2^s\right)\chi\left(\lambda/2^n\right)\,\mathrm{d}\lambda.
\end{aligned}$$

It is easy to check, with elementary computation, by using definition (7), that

$$\begin{aligned}\int_{-\infty}^{\infty} & e^{i\lambda(j/2^s-k/2^n)}\chi\left((\omega-\lambda)/2^s\right)\chi\left(\lambda/2^n\right)\,\mathrm{d}\lambda \\ &= \chi\left((\omega-2^{n+1}\pi)/2^s\right)\int_{2^{n+1}\pi}^{\omega-2^{s+1}\pi} e^{i\lambda(j/2^s-k/2^n)}\,\mathrm{d}\lambda \\ &+\chi\left((\omega-2^{s+2}\pi)/2^n\right)\int_{\omega-2^{s+2}\pi}^{2^{n+2}\pi} e^{i\lambda(j/2^s-k/2^n)}\,\mathrm{d}\lambda,\end{aligned}$$

i.e.,

$$\begin{aligned}\widehat{\psi_j^s(x)\,\psi_k^n}(x) = \chi\left((\omega-2^{n+1}\pi)/2^s\right) & \left.\frac{e^{i\lambda(j/2^s-k/2^n)}}{i(j/2^s-k/2^n)}\right|_{2^{n+1}\pi}^{\omega-2^{s+1}\pi} \\ &+\chi\left((\omega-2^{s+2}\pi)/2^n\right)\left.\frac{e^{i\lambda(j/2^s-k/2^n)}}{i(j/2^s-k/2^n)}\right|_{\omega-2^{s+2}\pi}^{2^{n+2}\pi}\end{aligned}$$

Thus, we have

$$\begin{aligned}\Xi_{jkh}^{snm} = \frac{2^{-(s+n+m)/2}}{4\pi^2}e^{-i\omega j/2^s}\Big\langle\chi\left((\omega-2^{n+1}\pi)/2^s\right) & \left.\frac{e^{i\lambda(j/2^s-k/2^n)}}{i(j/2^s-k/2^n)}\right|_{2^{n+1}\pi}^{\omega-2^{s+1}\pi} \\ +\chi\left((\omega-2^{s+2}\pi)/2^n\right) & \left.\frac{e^{i\lambda(j/2^s-k/2^n)}}{i(j/2^s-k/2^n)}\right|_{\omega-2^{s+2}\pi}^{2^{n+2}\pi}, e^{-i\omega k/2^m}\chi\left(\omega/2^m\right)\Big\rangle.\end{aligned}$$

It should be noticed that, even if it is not difficult the computation of the nonlinear connection coefficients, it is not possible to give a simple formula for their explicit form. However, in some cases, we can easily get a simple expression, for example,

$$\Xi_{000}^{000} = \frac{1}{4\pi^2}\left\langle\left[\chi\left(\omega-2\pi\right)-\chi\left(\omega-4\pi\right)\right]\left(\omega-4\pi\right),\chi\left(\omega\right)\right\rangle = 0,$$

analogously, we have $\Xi_{000}^{010}=0$, $\Xi_{000}^{001}=1/\sqrt{2}$, $\Xi_{000}^{011}\cong 0.249996$, $\Xi_{000}^{111}=0$. Up to the second scale, we have for Ξ_{000}^{0nm}, $(n=0,1,2)$, $(m=0,1,2)$,

m \ n	0	1	2
0	0	0	0
1	$\frac{1}{\sqrt{2}}$	0.249996	0
2	0	0	$\frac{5}{8}$

The second group of coefficients

$$\Gamma^{(\ell)}{}_{jkh}^{snm} = \left\langle\psi_j^s\frac{\mathrm{d}^\ell\psi_k^n}{\mathrm{d}x^\ell},\psi_h^m\right\rangle, \tag{37}$$

taking into account equations (14), (17), (18), is given by

$$\left\langle\psi_j^s(x)\frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell}\psi_k^n(x),\psi_h^m(x)\right\rangle = 2\pi\left\langle\psi_j^s(x)\widehat{\frac{\mathrm{d}^\ell\psi_k^n(x)}{\mathrm{d}x^\ell}},\widehat{\psi_h^m(x)}\right\rangle.$$

For the properties of the Fourier transform, with respect the convolution product, we have, for the first factor of the inner product,

$$\widehat{\psi_j^s(x)\frac{d^\ell\psi_k^n(x)}{dx^\ell}} = \widehat{\psi_j^s(x)} \star \widehat{\frac{d^\ell\psi_k^n(x)}{dx^\ell}} = \int_{-\infty}^{\infty}\hat{\psi}_j^s(\omega-\lambda)\widehat{\frac{d^\ell\psi_k^n}{dx^\ell}}(\lambda)\,d\lambda,$$

i.e., taking into account (17),

$$\widehat{\psi_j^s(x)\frac{d^\ell\psi_k^n(x)}{dx^\ell}} = \int_{-\infty}^{\infty}\hat{\psi}_j^s(\omega-\lambda)(i\lambda)^\ell\hat{\psi}_k^n(\lambda)\,d\lambda,$$

and (11),

$$\begin{aligned}\widehat{\psi_j^s(x)\frac{d^\ell\psi_k^n(x)}{dx^\ell}} &= \int_{-\infty}^{\infty}\frac{2^{-s/2}}{2\pi}e^{-i(\omega-\lambda)j/2^s}\chi\left((\omega-\lambda)/2^s\right)(i\lambda)^\ell\frac{2^{-n/2}}{2\pi}e^{-i\lambda k/2^n}\chi\left(\lambda/2^n\right)d\lambda,\\ &= \frac{2^{-s/2-n/2}}{4\pi^2}e^{-i\omega j/2^s}\int_{-\infty}^{\infty}(i\lambda)^\ell e^{i\lambda(j/2^s-k/2^n)}\chi\left((\omega-\lambda)/2^s\right)\chi\left(\lambda/2^n\right)d\lambda.\end{aligned}$$

It is easy to check, with elementary computation, by using definition (7), that

$$\begin{aligned}&\int_{-\infty}^{\infty}(i\lambda)^\ell e^{i\lambda(j/2^s-k/2^n)}\chi\left((\omega-\lambda)/2^s\right)\chi\left(\lambda/2^n\right)d\lambda\\ &= \chi\left((\omega-2^{n+1}\pi)/2^s\right)\int_{2^{n+1}\pi}^{\omega-2^{s+1}\pi}(i\lambda)^\ell e^{i\lambda(j/2^s-k/2^n)}\,d\lambda\\ &+\chi\left((\omega-2^{s+2}\pi)/2^n\right)\int_{\omega-2^{s+2}\pi}^{2^{n+2}\pi}(i\lambda)^\ell e^{i\lambda(j/2^s-k/2^n)}\,d\lambda,\end{aligned}$$

so that

$$\begin{aligned}\widehat{\psi_j^s(x)\frac{d^\ell\psi_k^n(x)}{dx^\ell}} &= \frac{2^{-s/2-n/2}}{4\pi^2}e^{-i\omega j/2^s}\times\\ &\left[\chi\left((\omega-2^{n+1}\pi)/2^s\right)\int_{2^{n+1}\pi}^{\omega-2^{s+1}\pi}(i\lambda)^\ell e^{i\lambda(j/2^s-k/2^n)}\,d\lambda\right.\\ &\left.+\chi\left((\omega-2^{s+2}\pi)/2^n\right)\int_{\omega-2^{s+2}\pi}^{2^{n+2}\pi}(i\lambda)^\ell e^{i\lambda(j/2^s-k/2^n)}\,d\lambda\right].\end{aligned}$$

Although the two integrals, in brackets, depend on the dilation-translation parameters n, s, k, j, they can be easily estimated. For the first integral, it is

$$\begin{aligned}0 &\le \chi\left((\omega-2^{n+1}\pi)/2^s\right)\int_{2^{n+1}\pi}^{\omega-2^{s+1}\pi}(i\lambda)^\ell e^{i\lambda(j/2^s-k/2^n)}\,d\lambda\\ &< \int_0^{2^{s+1}\pi}(i\lambda)^\ell e^{i\lambda(j/2^s-k/2^n)}\,d\lambda,\end{aligned}$$

and analogously, for the second

$$\int_{2^{n+1}\pi}^{2^{n+2}\pi}(i\lambda)^\ell e^{i\lambda(j/2^s-k/2^n)}\,d\lambda \le \chi\left((\omega-2^{s+2}\pi)/2^n\right)\int_{\omega-2^{s+2}\pi}^{2^{n+2}\pi}(i\lambda)^\ell e^{i\lambda(j/2^s-k/2^n)}\,d\lambda < 0.$$

It does not seem to be possible to get a general formula like (27) also for the coefficients $(35)_2$. If we assume $\ell = 1$, we have

$$\begin{aligned}\widehat{\psi_j^s(x)\frac{d\psi_k^n(x)}{dx}} &= \frac{2^{-s/2-n/2}}{4\pi^2}e^{-i\omega j/2^s}\times\\ &\left[\chi\left((\omega-2^{n+1}\pi)/2^s\right)\int_{2^{n+1}\pi}^{\omega-2^{s+1}\pi}(i\lambda)e^{i\lambda(j/2^s-k/2^n)}\,d\lambda\right.\\ &\left.+\chi\left((\omega-2^{s+2}\pi)/2^n\right)\int_{\omega-2^{s+2}\pi}^{2^{n+2}\pi}(i\lambda)e^{i\lambda(j/2^s-k/2^n)}\,d\lambda\right],\end{aligned}$$

and a necessary condition that the inner product with

$$\widehat{\psi_h^m(x)} = \frac{2^{-m/2}}{2\pi} e^{-i\omega h/2^m} \chi(\omega/2^m),$$

be different from zero is $\chi\left((\omega - 2^{n+1}\pi)/2^s\right)\chi(\omega/2^m) \neq 0$ or $\chi\left((\omega - 2^{s+2}\pi)/2^n\right)\chi(\omega/2^m) \neq 0$, and this, namely, depends on the scaling parameters (it can be easily seen that when $n = m = s$ they vanish). With a direct (numerical) computation we have, e.g.,

$$\Gamma^{000}_{000} = \Gamma^{010}_{000} = \Gamma^{100}_{000} = \Gamma^{110}_{000} = \Gamma^{111}_{000} \cong 0, \qquad \Gamma^{001}_{000} \cong 6.66436i, \qquad \Gamma^{011}_{000} \cong 3.66511i.$$

5. GALERKIN APPROXIMATION OF DIFFERENTIAL OPERATORS IN WAVELET SPACES

In this section, a classical Petrov-Galerkin method to approximate evolution operators in wavelet space (see, e.g., the pioneering papers [4–6] and references therein) is briefly sketched. Let us take the classical problem,

$$\frac{\partial u}{\partial t} = Lu, \tag{38}$$

where $L = L\left(\frac{\partial}{\partial x}, \frac{\partial^2}{\partial x^2}, \dots\right)$. Let us assume the solution to be the real function,

$$u(x,t) = \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} \beta_k^n(t)\psi_k^n(x),$$

with $\beta_k^n(t) \stackrel{\text{def}}{=} \langle u(x,t), \psi_k^n(x)\rangle$ and Π^N be the projection into the N-scale approximation wavelet space, i.e.,

$$\Pi^N u(x,t) = \sum_{n=0}^{N} \sum_{k=0}^{2^n-1} \beta_k^n(t)\psi_k^n(x).$$

The projection of the evolution operator into the N-scale space depends on its specific form. For instance for a linear operator,

$$L = a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial^2}{\partial x^2} + \cdots = \sum_{\ell=1}^{p} a_\ell \frac{\partial^\ell}{\partial x^\ell},$$

it is

$$\begin{aligned}\Pi^N \left[\sum_{\ell=1}^{p} a_\ell \frac{\partial^\ell}{\partial x^\ell} u(x,t)\right] &= \sum_{n=0}^{N} \sum_{k=0}^{2^n-1} \beta_k^n(t) \left\{\Pi^N \left[\sum_{\ell=1}^{p} a_\ell \frac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} \psi_k^n(x)\right]\right\} \\ &= \sum_{n=0}^{N} \sum_{k=0}^{2^n-1} \beta_k^n(t) \sum_{\ell=1}^{p} a_\ell \gamma^{(\ell)}{}^{nm}_{kh} \psi_h^m(x),\end{aligned}$$

and the evolution problem (38) becomes

$$\sum_{n=0}^{N} \sum_{k=0}^{2^n-1} \frac{\mathrm{d}\beta_k^n(t)}{\mathrm{d}t} \psi_k^n(x) = \sum_{n=0}^{N} \sum_{k=0}^{2^n-1} \beta_k^n(t) \sum_{\ell=1}^{p} a_\ell \gamma^{(\ell)}{}^{nm}_{kh} \psi_h^m(x)$$

i.e., taking into account the orthogonality condition (15),

$$\frac{\mathrm{d}\beta_h^m(t)}{\mathrm{d}t} = \sum_{n=0}^{N} \sum_{k=0}^{2^n-1} \beta_k^n(t) \sum_{\ell=1}^{p} a_\ell \gamma^{(\ell)}{}^{nm}_{kh} \qquad (m = 0, \dots, N;\ h = 0, \dots, 2^m - 1).$$

For a quadratic operator,

$$L = a_1 u \frac{\partial}{\partial x} + a_2 u \frac{\partial^2}{\partial x^2} + \dots,$$

in the last formula there appear the connection coefficients $\Gamma^{(\ell)}{}^{nm}_{kh}$, and so on for more general operators.

6. CONCLUSION

The application of the Galerkin method for the approximation of solution of PDE depends on the easy treatment of the connection coefficients. In this paper, a finite formula for a simple computation of a class of connection coefficients of harmonic wavelets has been given for any order of derivatives. Some of their recursive properties were also given. As a first step towards the solution of evolution problems, it has been given the explicit computation of some connection coefficients and it was discussed the generalization to some nonlinear cases. The wavelet Petrov-Galerkin method was shortly sketched.

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